Electrodynamics of Moving Media

Yasuyoshi Horibata

Institute of Space and Aeronautical Science, University of Tokyo

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On the basis of the Minkowski formulation, the total energy-momentum tensor of a system consisting of matter and electromagnetic fields is derived from the macroscopic theory. The analysis of this tensor shows that the electromagnetic fields supply the matter with momentum and energy. Consequently, the electromagnetic part and the material part overlap each other in the total energy-momentum tensor. Hence it is impossible to divide the total energy-momentum tensor into an electromagnetic tensor and a material tensor.

In a closed system, in general, only the total energy-momentum tensor has physical significance and can be defined.

Further, the generalized force which acts on the matter is obtained and interpreted clearly.

I. Introduction

Electrodynamics of moving media has been discussed by a large number of investigators for long years, and until the present many different forms of an electromagnetic energy-momentum tensor within media have been put forward [1–4]. However, no form is universally accepted, although Minkowski's tensor is the most well-known and widely used of them [5].

The present paper develops Møller's theory of an elastic body [6] and analyzes, on the basis of the Minkowski formulation, a system consisting of matter and electromagnetic fields. The effect of the electromagnetic fields on the matter is revealed. Further, the generalized force which acts on the matter is obtained and interpreted clearly.

For the sake of simplicity, only isotropic and nondispersive media with linear constitutive relations are considered. Further, the summation convention is used. The Latin subscripts assume the values 1, 2, 3, 4, whereas the Greek subscripts assume the values 1, 2, 3.

II. The Lorentz Force, the Joule Heat, and the Electrostriction and Magnetostriction Forces

In the first place, the force per unit volume that acts on stationary media in stationary electromagnetic fields is considered. If the elastic force is left out of consideration, it is given by [7, 8]

$$\boldsymbol{f}_{\text{stat}} = \varrho\,\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B} - \tfrac{1}{2}E^2 \operatorname{grad}\varepsilon - \tfrac{1}{2}H^2 \operatorname{grad}\mu + \operatorname{div}{}^2\sigma\,, \qquad (1)$$

Reprint requests to Yasuyoshi Horibata, Institute of Space and Aeronautical Science, University of Tokyo, Komaba, Meguro-ku, Tokyo, Japan.

where ϱ and $\boldsymbol{J}$ are the true charge and current densities. The last term in (1) is the vector whose components are

$$(\operatorname{div}{}^2\sigma)_\alpha = \partial\sigma_{\alpha\beta}/\partial x_\beta\,, \qquad (2)$$

where

$$\sigma_{\alpha\beta} = -\tfrac{1}{2}a_1 E_\alpha E_\beta - \tfrac{1}{2}a_2 E^2\delta_{\alpha\beta} - \tfrac{1}{2}b_1 H_\alpha H_\beta - \tfrac{1}{2}b_2 H^2\delta_{\alpha\beta}\,; \qquad (3)$$

the Kronecker symbol is denoted by $\delta_{\alpha\beta}$. The coefficients a_1 and a_2 in (3) represent the rate of change due to the strains of the permittivity ε of the medium, whereas b_1 and b_2 represent that of the permeability μ.

Note that the last term in (1) is quite different from the other terms in nature. It represents the electrostriction and magnetostriction forces; these act as surface forces, because $\sigma_{\alpha\beta}$ represents the internal stresses which occur to evoke the strains. In contrast with it, the first four terms in (1) represent the Lorentz force, which acts as a volume force.

Even in the general case of time-dependent fields, only the Lorentz force and the electrostriction and magnetostriction forces act immediately on the medium. Moreover if the fields do not vary rapidly, it may be assumed that the Lorentz force and the electrostriction and magnetostriction forces are represented by (1). The generalized force is discussed in detail in Section VI.

Next the case in which media are moving with a constant velocity $\boldsymbol{v}$ is considered. The following identity is generated from Minkowski's field equations by the tensor manipulation [9]:

$$f_i^* = -\partial S_{ik}/\partial x_k\,, \qquad (4)$$

where

$$f_i^* = F_{il}J_l + \frac{1}{4}\left(F_{kl}\frac{\partial H_{kl}}{\partial x_i} - \frac{\partial F_{kl}}{\partial x_i}H_{kl}\right), \qquad (5)$$

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and where

$$S_{ik}=F_{il}H_{kl}-\tfrac{1}{2}\delta_{ik}(F_{lm}H_{lm}) \tag{6}$$

is Minkowski's tensor. The antisymmetrical field tensors F_{ik} and H_{ik} are defined by

$$(F_{23},F_{31},F_{12})=c\,\boldsymbol{B}\,,\quad (F_{41},F_{42},F_{43})=i\,\boldsymbol{E}\,, \tag{7}$$

$$(H_{23},H_{31},H_{12})=\boldsymbol{H}/c\,,\quad (H_{41},H_{42},H_{43})=i\,\boldsymbol{D}\,, \tag{8}$$

and

$$J_i=(\boldsymbol{J}/c,i\,\varrho) \tag{9}$$

is the four-current density.

The four-vector f_i^* can be divided into the two four-vectors f_i and π_i [10], i. e.,

$$f_i^*=f_i+\pi_i\,; \tag{10}$$

these f_i and π_i are of form

$$f_i=\left\{\mathbf{f},\ \frac{i}{c}(\mathbf{f}\cdot\boldsymbol{v})\right\}, \tag{11}$$

$$\pi_i=\left\{\frac{\varphi\,\boldsymbol{v}}{c^2},\ \frac{i}{c}\varphi\right\}. \tag{12}$$

If the superscript 0 henceforth denotes all field quantities evaluated in the rest inertial frame S^0, in which the velocity of the medium at the considered space-time point is momentarily zero, then $\mathbf{f}^0$ and φ^0 become, from (5),

$$\mathbf{f}^0=\varrho^0\,\boldsymbol{E}^0+\boldsymbol{J}^0\times\boldsymbol{B}^0-\tfrac{1}{2}E^{02}\,\mathrm{grad}^0\,\varepsilon-\tfrac{1}{2}H^{02}\,\mathrm{grad}^0\,\mu\,, \tag{13}$$

$$\varphi^0=\boldsymbol{E}^0\cdot\boldsymbol{J}^0\,. \tag{14}$$

The Lorentz force is a mechanical force, i. e., the time component of its four-force density always vanishes in S^0. Hence, from (1), (11), and (13), it follows that f_i is the four-force density of the Lorentz force. As for π_i, according to Møller [11], it represents the momentum and energy of the medium produced by the Joule heat per unit time and volume.

Thus the action of the Lorentz force and the Joule heat on the medium can be described in terms of f_i^*; this is equal to the negative divergence of Minkowski's tensor.

III. Derivation of the Total Energy-Momentum Tensor

By the application of the fundamental equations of mechanics, this section derives the total energy-momentum tensor of a system consisting of electromagnetic fields and media.

In general, the media are deforming and accelerating. However, it is assumed that the strains are small enough for the change of ε and μ to be neglected and, further, that in S^0 for the medium at a considered space-time point Maxwell's equations for stationary media hold in the neighbourhood of this medium. Finally, to simplify matters, heat conduction in the media is assumed to be negligible.

In addition to the Lorentz force and the electrostriction and magnetostriction forces, the elastic force exists in the media. It acts as a surface force like the electrostriction and magnetostriction forces. These two kinds of surface forces are dealt with together, so that the surface force acting through a face element $\mathrm{d}f$ is written as

$$\mathrm{d}\boldsymbol{t}(\boldsymbol{n})=\boldsymbol{t}(\boldsymbol{n})\,\mathrm{d}f\,, \tag{15}$$

where $\boldsymbol{n}$ is a unit vector normal to $\mathrm{d}f$, and $\mathrm{d}\boldsymbol{t}$ is the force acting from the side to which $\boldsymbol{n}$ points. The components $t_\alpha(\boldsymbol{n})$ of $\boldsymbol{t}(\boldsymbol{n})$ can be written in terms of the components n_α of $\boldsymbol{n}$ as

$$t_\alpha(\boldsymbol{n})=t_{\alpha\beta}\,n_\beta\,, \tag{16}$$

where

$$t_{\alpha\beta}=\tau_{\alpha\beta}+\sigma_{\alpha\beta}\,. \tag{17}$$

This $\tau_{\alpha\beta}$ is the elastic stress tensor; in S^0 this is connected with the strains by means of the usual theory of elasticity. On the other hand, $\sigma_{\alpha\beta}$ is the electromagnetic stress tensor; in S^0 this is given by (3).

The total surface force $\boldsymbol{F}$ acting on the media inside a closed surface f can be written, by the application of Gauss's theorem, as

$$F_\alpha=\int_f t_{\alpha\beta}\,n_\beta\,\mathrm{d}f=\int_V(\partial t_{\alpha\beta}/\partial x_\beta)\,\mathrm{d}V\,, \tag{18}$$

where V is the volume surrounded by f. Hence if both the Lorentz force and the Joule heat also are taken into consideration, the equation of motion for a volume element δV is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}(g_\alpha\,\delta V)=\frac{\partial t_{\alpha\beta}}{\partial x_\beta}\,\delta V+f_\alpha^*\,\delta V\,, \tag{19}$$

where $\boldsymbol{g}$ is the momentum density of the medium, and $\mathrm{d}/\mathrm{d}t$ denotes the substantial time derivative. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(g_\alpha\,\delta V)=\left\{\frac{\partial g_\alpha}{\partial t}+\frac{\partial(g_\alpha u_\beta)}{\partial x_\beta}\right\}\delta V\,, \tag{20}$$

substitution of (4) and (20) into (19) yields

$$\frac{\partial}{\partial t}\left(g_\alpha+\frac{1}{i\,c}S_{\alpha 4}\right)+\frac{\partial}{\partial x_\beta}(g_\alpha u_\beta-t_{\alpha\beta}+S_{\alpha\beta})=0\,, \tag{21}$$

where $\boldsymbol{u}(x_i)$ is the velocity of the medium.

The total work which the surface forces do on the media inside f per unit time is given by

$$A = \int_f t_{\alpha\beta}\, n_\beta\, u_\alpha\, \mathrm{d}f = \int_V \frac{\partial}{\partial x_\beta} (t_{\alpha\beta}\, u_\alpha)\, \mathrm{d}V\,. \quad (22)$$

Accordingly, the energy balance for δV can be expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t} (h\, \delta V) = \frac{\partial}{\partial x_\beta} (t_{\alpha\beta}\, u_\alpha)\, \delta V + \frac{c}{i} f_4{}^* \,\delta V\,, \quad (23)$$

where h is the energy density of the medium. By arguments similar to those used before, (23) reduces to

$$\frac{\partial}{\partial t} (h - S_{44}) + \frac{\partial}{\partial x_\beta} \left(h\, u_\beta - t_{\alpha\beta}\, u_\alpha + \frac{c}{i} S_{4\beta} \right) = 0\,. \quad (24)$$

Equations (21) and (24) represent the conservation laws of momentum and energy respectively.

If, on the other hand, T_{ik} denotes the total energy-momentum tensor, the conservation laws can be written in the simple form [12]

$$\partial T_{ik}/\partial x_k = 0\,; \quad (25)$$

this T_{ik} must be symmetrical, i. e.,

$$T_{ik} = T_{ki}\,. \quad (26)$$

Equation (25) for $i = 1, 2, 3$ represents the conservation of momentum and for $i = 4$ represents that of energy. By comparing (21) and (24) with (25) and (26), therefore, the components of T_{ik} are obtained:

$$T_{\alpha\beta} = g_\alpha\, u_\beta - t_{\alpha\beta} - s_{\alpha\beta}\,, \quad (27)$$

where

$$g_\alpha = \frac{1}{c^2} \{ h\, u_\alpha - t_{\gamma\alpha}\, u_\gamma + (\mathbf{E} \times \mathbf{H} - c^2\, \mathbf{D} \times \mathbf{B})_\alpha \}\,, \quad (28)$$

$$s_{\alpha\beta} = E_\alpha\, D_\beta + H_\alpha\, B_\beta - \tfrac{1}{2} (\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\, \delta_{\alpha\beta}\,; \quad (29)$$

further,

$$(T_{41}, T_{42}, T_{43}) = \frac{i}{c} \mathbf{S}\,, \quad (30)$$

where

$$\mathbf{S} = h\, \boldsymbol{u} - \boldsymbol{u} * \boldsymbol{t} + \mathbf{E} \times \mathbf{H} \quad (31)$$

is the total energy current density, and

$$(T_{14}, T_{24}, T_{34}) = i\, c\, \mathbf{G}\,, \quad (32)$$

where

$$\mathbf{G} = \frac{1}{c^2} (h\, \boldsymbol{u} - \boldsymbol{u} * \boldsymbol{t} + \mathbf{E} \times \mathbf{H}) \quad (33)$$

is the total momentum density; finally,

$$T_{44} = -H\,, \quad (34)$$

where

$$H = h + \tfrac{1}{2} (\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}) \quad (35)$$

is the total energy density. In the above equations, $\boldsymbol{u} * \boldsymbol{t}$ denotes the vector whose components are

$$(\boldsymbol{u} * \boldsymbol{t})_\alpha = u_\beta\, t_{\beta\alpha}\,. \quad (36)$$

Next the tensor R_{ik} is introduced by

$$T_{ik} = R_{ik} + S_{ik}\,. \quad (37)$$

From (27) – (37), its components are obtained:

$$R_{\alpha\beta} = g_\alpha\, u_\beta - t_{\alpha\beta}\,, \quad (38)$$

$$(R_{41}, R_{42}, R_{43}) = \frac{i}{c} \boldsymbol{s} = \frac{i}{c} (h\, \boldsymbol{u} - \boldsymbol{u} * \boldsymbol{t})\,, \quad (39)$$

$$(R_{14}, R_{24}, R_{34}) = i\, c\, \boldsymbol{g} = \frac{i}{c} \cdot (h\, \boldsymbol{u} - \boldsymbol{u} * \boldsymbol{t} + \mathbf{E} \times \mathbf{H} - c^2\, \mathbf{D} \times \mathbf{B})\,, \quad (40)$$

$$R_{44} = -h\,. \quad (41)$$

IV. Consideration of the Surface Force $\mathrm{d}\boldsymbol{t}(\boldsymbol{n})$

This section proves that the surface force $\mathrm{d}\boldsymbol{t}(\boldsymbol{n})$ is a mechanical force, i. e., that

$$\left\{ \gamma\, \mathrm{d}\boldsymbol{t}(\boldsymbol{n}),\ \frac{i}{c} \gamma\, [\mathrm{d}\boldsymbol{t}(\boldsymbol{n}) \cdot \boldsymbol{u}] \right\} \quad (42)$$

is a four-vector. Here

$$\gamma = 1 \big/ \sqrt{1 - u^2/c^2}\,. \quad (43)$$

First the following scalar is obtained from R_{ik}:

$$U_i\, R_{ik}\, U_k/c^2 = U_i{}^0\, R_{ik}{}^0\, U_k{}^0/c^2 = h^0(x_i)\,, \quad (44)$$

where

$$U_i = (\gamma\, \boldsymbol{u}, i\, c\, \gamma) \quad (45)$$

is the four-velocity. Note that $h^0(x_i)$ is the rest energy density of the medium. In terms of the total energy density H^0 in S^0, this h^0 is determined by (35). Then the two tensors Θ_{ik} and Q_{ik} are introduced by

$$\Theta_{ik} = \mu^0\, U_i\, U_k\,, \quad (46)$$

$$R_{ik} = \Theta_{ik} + Q_{ik}\,, \quad (47)$$

where

$$\mu^0 = h^0/c^2 \quad (48)$$

is the rest mass density of the medium. Further, p_{ik} is defined by

$$p_{ik} = R_{ik} - R_{i4}\, U_k/U_4 = Q_{ik} - Q_{i4}\, U_k/U_4\,. \quad (49)$$

Note that p_{ik} is not a tensor, because the components p_{i4} vanish in any inertial frame, as is easily seen from (49).

The components $p_{\alpha\beta}$ can be evaluated with the aid of (38), (40), and (49), so that

$$p_{\alpha\beta} = R_{\alpha\beta} - R_{\alpha 4}\, U_\beta / U_4 = -t_{\alpha\beta}\,. \tag{50}$$

By using the equation

$$U_i\, R_{ik} = -h^0\, U_k\,, \tag{51}$$

which follows from the fact that $U_i^0\, R_{ik}^0 = -h^0\, U_k^0$, $U_i\, p_{ik}$ becomes

$$U_i\, p_{ik} = U_i\, R_{ik} - U_i\, R_{i4}\, U_k / U_4 = 0\,. \tag{52}$$

Hence, from (52), the components $p_{4\alpha}$ are evaluated as

$$p_{4\alpha} = -U_\beta\, p_{\beta\alpha}/U_4 = -\frac{i}{c}\, u_\beta\, t_{\beta\alpha}\,. \tag{53}$$

With the substitution of (50) and (53), $p_{i\beta}\, n_\beta\, \mathrm{d}f$ becomes

$$\begin{aligned} p_{i\beta}\, n_\beta\, \mathrm{d}f &= (p_{\alpha\beta}\, n_\beta\, \mathrm{d}f,\, p_{4\beta}\, n_\beta\, \mathrm{d}f) \\ &= -\left\{\mathrm{d}\boldsymbol{t}(\boldsymbol{n}),\ \frac{i}{c}\,[\mathrm{d}\boldsymbol{t}(\boldsymbol{n})\cdot\boldsymbol{u}]\right\}. \end{aligned} \tag{54}$$

Since $\mathrm{d}F_i$ defined by

$$\mathrm{d}F_i = \left(\gamma\, n_\alpha\, \mathrm{d}f,\ \frac{i}{c}\,\gamma\, u_\beta\, n_\beta\, \mathrm{d}f\right) \tag{55}$$

is a four-vector [13], the four-vector $\mathrm{d}T_i$ can be introduced by

$$\mathrm{d}T_i = -R_{ik}\, \mathrm{d}F_k\,. \tag{56}$$

This $\mathrm{d}T_i$ becomes, from (49), (54), (55), and (56),

$$\begin{aligned} \mathrm{d}T_i &= -\gamma\left(R_{i\beta} + \frac{i}{c}\, R_{i4}\, u_\beta\right) n_\beta\, \mathrm{d}f = -\gamma\, p_{i\beta}\, n_\beta\, \mathrm{d}f \\ &= \left\{\gamma\, \mathrm{d}\boldsymbol{t}(\boldsymbol{n}),\ \frac{i}{c}\,\gamma[\mathrm{d}\boldsymbol{t}(\boldsymbol{n})\cdot\boldsymbol{u}]\right\}\,; \end{aligned} \tag{57}$$

this is equal to (42).

V. The Interaction Between the Electromagnetic Fields and the Media

If the transformation law of a tensor is applied to R_{ik}, then $t_{\alpha\beta}$, $\boldsymbol{g}$, and h can be expressed in terms of the components of R_{ik}^0 in S^0. The results are as follows:

$$t_{\alpha\beta} = t^0_{\alpha\beta} + \frac{\gamma-1}{u^2}\, u_\alpha\, u_\lambda\, t^0_{\lambda\beta} - \frac{\gamma-1}{u^2\,\gamma}\, t^0_{\alpha\lambda}\, u_\lambda\, u_\beta - \frac{(\gamma-1)^2}{u^4\,\gamma}\, u_\alpha\, u_\beta\, u_\lambda\, t^0_{\lambda\eta}\, u_\eta\,, \tag{58}$$

$$g_\alpha = \frac{\gamma^2}{c^2}\left[h^0 - \frac{1-\gamma^{-1}}{u^2}\,\{u_\lambda\, t^0_{\lambda\eta}\, u_\eta - (\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\,\boldsymbol{D}^0\times\boldsymbol{B}^0)\cdot\boldsymbol{u}\}\right] u_\alpha - \frac{\gamma}{c^2}\,\{t^0_{\alpha\lambda}\, u_\lambda - (\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\,\boldsymbol{D}^0\times\boldsymbol{B}^0)_\alpha\}\,, \tag{59}$$

$$h = \gamma^2\left[h^0 - \frac{1}{c^2}\,\{u_\lambda\, t^0_{\lambda\eta}\, u_\eta - (\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\,\boldsymbol{D}^0\times\boldsymbol{B}^0)\cdot\boldsymbol{u}\}\right], \tag{60}$$

$$s_\alpha = \gamma^2\left\{h^0 - \frac{1-\gamma^{-1}}{u^2}\, u_\lambda\, t^0_{\lambda\eta}\, u_\eta + \frac{1}{c^2}\,(\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\,\boldsymbol{D}^0\times\boldsymbol{B}^0)\cdot\boldsymbol{u}\right\} u_\alpha - \gamma\, u_\lambda\, t^0_{\lambda\alpha}\,. \tag{61}$$

These equations show that $t_{\alpha\beta}$ is determined only by $t^0_{\alpha\beta}$, whereas both h and $\boldsymbol{g}$ depend not only on h^0, but also on $t^0_{\alpha\beta}$, E_α^0, D_α^0, H_α^0, and B_α^0. Hence the energy and momentum of the medium depend explicitly on the elastic stresses and the electromagnetic fields.

Equations (37), (59), (60), and (61) give

$$\begin{aligned} G_\alpha = \frac{\gamma^2}{c^2}\left[h^0 - \frac{1-\gamma^{-1}}{u^2}\,\{u_\lambda\, t^0_{\lambda\eta}\, u_\eta - (\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\,\boldsymbol{D}^0\times\boldsymbol{B}^0)\cdot\boldsymbol{u}\}\right] u_\alpha \\ - \frac{\gamma}{c^2}\,\{t^0_{\alpha\lambda}\, u_\lambda - (\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\,\boldsymbol{D}^0\times\boldsymbol{B}^0)_\alpha\} + (\boldsymbol{D}\times\boldsymbol{B})_\alpha\,, \end{aligned} \tag{62}$$

$$H = \gamma^2\left[h^0 - \frac{1}{c^2}\,\{u_\lambda\, t^0_{\lambda\eta}\, u_\eta - (\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\,\boldsymbol{D}^0\times\boldsymbol{B}^0)\cdot\boldsymbol{u}\}\right] + \frac{1}{2}\,(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})\,, \tag{63}$$

$$S_\alpha = \gamma^2\left\{h^0 - \frac{1-\gamma^{-1}}{u^2}\, u_\lambda\, t^0_{\lambda\eta}\, u_\eta + \frac{1}{c^2}\,(\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\,\boldsymbol{D}^0\times\boldsymbol{B}^0)\cdot\boldsymbol{u}\right\} u_\alpha - \gamma\, u_\lambda\, t^0_{\lambda\alpha} + (\boldsymbol{E}\times\boldsymbol{H})_\alpha\,, \tag{64}$$

where

$$t^0_{\alpha\beta} = \tau^0_{\alpha\beta} - \tfrac{1}{2}\, a_1\, E_\alpha^0\, E_\beta^0 - \tfrac{1}{2}\, a_2\, E^{02}\, \delta_{\alpha\beta} - \tfrac{1}{2}\, b_1\, H_\alpha^0\, H_\beta^0 - \tfrac{1}{2}\, b_2\, H^{02}\, \delta_{\alpha\beta}\,. \tag{65}$$

According to Minkowski, the last terms in (62), (63), and (64) are the electromagnetic momentum density, energy density, and energy current density respectively. In addition to the last terms, however, the terms exist involving $E_a{}^0$, $D_a{}^0$, $H_a{}^0$, or $B_a{}^0$. These terms represent the interaction between the electromagnetic fields and the media. When the electromagnetic momentum density, energy density, and energy current density come into question, these terms also should be included in them. Therefore, in the total energy-momentum tensor, the electromagnetic part and the material part overlap each other.

Next the problem of negative energy [14] is considered. Let a homogeneous insulator with an index of refraction n move with a constant velovity $\bar{\boldsymbol{u}} = (\bar{u}, 0, 0)$. If a plane electromagnetic wave travels along the positive $x_1{}^0$-axis within the body in S^0, then

$$\tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{H}\cdot\boldsymbol{B}) = \bar{\gamma}^2(1+\beta/n)(1+n\beta)\,\varepsilon E^{02}, \quad (66)$$

where

$$\beta = \bar{u}/c, \quad \bar{\gamma} = 1/\sqrt{1-\bar{u}^2/c^2}. \quad (67)$$

When $\bar{u} < -c/n$, this electromagnetic energy density of Minkowski's becomes negative.

This problem can be solved in the following way. If the field $\boldsymbol{E}^0$ of the wave is assumed to vary very gradually in space in S^0, the body is nearly in static equilibrium in S^0. In the present case, moreover, no Lorentz force exists, so that $\tau_{11}^0 + \sigma_{11}^0 = 0$. Hence the total energy density becomes, from (63),

$$H = \bar{\gamma}^2 h^0 + \frac{\bar{\gamma}^2}{c^2}(\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\boldsymbol{D}^0\times\boldsymbol{B}^0)_1\,\bar{u} + \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{H}\cdot\boldsymbol{B}). \quad (68)$$

Adding the second and third terms in (68), one obtains

$$\frac{\bar{\gamma}^2}{c^2}(\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\boldsymbol{D}^0\times\boldsymbol{B}^0)_1\,\bar{u} + \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{H}\cdot\boldsymbol{B}) = \bar{\gamma}^2\{(\beta+1/n)^2+1-1/n^2\}\,\varepsilon E^{02}; \quad (69)$$

this always becomes positive. Hence the total energy density also is always positive. Note that (69) is not the electromagnetic energy density itself, because (69) does not include the term involving σ_{11}^0.

Since Minkowski's tensor comprises only part of the effect that is produced by the electromagnetic fields, such a curious phenomenon takes place.

VI. The Generalized Four-Force Density

This section makes clearer the interaction between the electromagnetic fields and the media.

If the four-vector W_i is introduced by

$$W_i = Q_{ik}U_k = (R_{ik} - \Theta_{ik})U_k, \quad (70)$$

Eqs. (40), (41), and (46) give

$$W_i^0 = \{-(\boldsymbol{E}^0\times\boldsymbol{H}^0 - c^2\boldsymbol{D}^0\times\boldsymbol{B}^0), 0\}. \quad (71)$$

Hence the components of W_i become, from (71),

$$W_i = \{-c^2\gamma^{-1}\boldsymbol{K}, \ -ic\gamma^{-1}(\boldsymbol{K}\cdot\boldsymbol{u})\}, \quad (72)$$

where

$$\boldsymbol{K} = \frac{\boldsymbol{E}\times\boldsymbol{H} - c^2\boldsymbol{D}\times\boldsymbol{B} + \boldsymbol{u}\times(\boldsymbol{D}\times\boldsymbol{E}+\boldsymbol{B}\times\boldsymbol{H})}{c^2-u^2}. \quad (73)$$

Further, multiplication of U_k and (49) leads to

$$Q_{i4}/U_4 = (p_{ik}U_k - W_i)/c^2, \quad (74)$$

so that Q_{ik} can be written as

$$Q_{ik} = p_{ik} + Q_{i4}U_k/U_4 = p_{ik} + p_{im}U_mU_k/c^2 - W_iU_k/c^2. \quad (75)$$

Hence, with the aid of (46), (50), (53), (72), and (75), the components of $T_{ik} = \Theta_{ik} + Q_{ik} + S_{ik}$ can be rewritten as

$$T_{\alpha\beta} = \frac{h^0}{c^2-u^2}u_\alpha u_\beta - t_{\alpha\beta} - \frac{t_{\alpha\lambda}u_\lambda}{c^2-u^2}u_\beta + K_\alpha u_\beta - \left\{E_\alpha D_\beta + H_\alpha B_\beta - \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{H}\cdot\boldsymbol{B})\,\delta_{\alpha\beta}\right\}, \quad (76)$$

$$G_\alpha = \frac{h^0}{c^2-u^2}u_\alpha - \frac{t_{\alpha\lambda}u_\lambda}{c^2-u^2} + K_\alpha + (\boldsymbol{D}\times\boldsymbol{B})_\alpha, \quad (77)$$

$$H = \frac{h^0c^2}{c^2-u^2} - \frac{u_\beta t_{\beta\lambda}u_\lambda}{c^2-u^2} + \boldsymbol{K}\cdot\boldsymbol{u} + \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{H}\cdot\boldsymbol{B}), \quad (78)$$

$$S_\alpha = \frac{h^0c^2}{c^2-u^2}u_\alpha - u_\beta t_{\beta\alpha} - \frac{u_\beta t_{\beta\lambda}u_\lambda}{c^2-u^2}u_\alpha + (\boldsymbol{K}\cdot\boldsymbol{u})\,u_\alpha + (\boldsymbol{E}\times\boldsymbol{H})_\alpha. \quad (79)$$

These equations show that both the electromagnetic fields and the elastic stresses supply the medium with momentum and energy; these travel with the medium. The densities of the supplied momentum and energy are

$$-t_{\alpha\lambda}u_\lambda/(c^2-u^2) + K_\alpha \quad (80)$$

and

$$-u_\beta t_{\beta\lambda}u_\lambda/(c^2-u^2) + \boldsymbol{K}\cdot\boldsymbol{u} \quad (81)$$

respectively.

The conservation laws (25) can be transformed into the form

$$\partial \Theta_{ik}/\partial x_k = f_i^+ , \quad (82)$$

with

$$f_i^+ = -\frac{\partial Q_{ik}}{\partial x_k} - \frac{\partial S_{ik}}{\partial x_k} . \quad (83)$$

By substituting (4) and (75) into (83), the components of $f_i^+ = (\mathbf{f}^+, f_4^+)$ are obtained:

$$f_a^+ = \varrho E_a + (\boldsymbol{J} \times \boldsymbol{B})_a + \frac{1}{2}\left(\frac{\partial \boldsymbol{E}}{\partial x_a}\cdot \boldsymbol{D} - \boldsymbol{E}\cdot \frac{\partial \boldsymbol{D}}{\partial x_a} + \frac{\partial \boldsymbol{H}}{\partial x_a}\cdot \boldsymbol{B} - \boldsymbol{H}\cdot \frac{\partial \boldsymbol{B}}{\partial x_a}\right)$$
$$+ \frac{\partial t_{a\beta}}{\partial x_\beta} + \frac{\partial}{\partial x_\beta}\left(\frac{t_{a\lambda}\, u_\lambda}{c^2 - u^2}\, u_\beta\right) + \frac{\partial}{\partial t}\left(\frac{t_{a\lambda}\, u_\lambda}{c^2 - u^2}\right) - \frac{\partial}{\partial x_\beta}(K_a\, u_\beta) - \frac{\partial K_a}{\partial t}, \quad (84)$$

$$\frac{c}{i} f_4^+ = \boldsymbol{E}\cdot\boldsymbol{J} - \frac{1}{2}\left(\frac{\partial \boldsymbol{E}}{\partial t}\cdot \boldsymbol{D} - \boldsymbol{E}\cdot \frac{\partial \boldsymbol{D}}{\partial t} + \frac{\partial \boldsymbol{H}}{\partial t}\cdot \boldsymbol{B} - \boldsymbol{H}\cdot \frac{\partial \boldsymbol{B}}{\partial t}\right)$$
$$+ \frac{\partial}{\partial x_a}(u_\beta\, t_{\beta a}) + \frac{\partial}{\partial x_a}\left(\frac{u_\beta\, t_{\beta\lambda}\, u_\lambda}{c^2 - u^2}\, u_a\right) + \frac{\partial}{\partial t}\left(\frac{u_\beta\, t_{\beta\lambda}\, u_\lambda}{c^2 - u^2}\right) - \frac{\partial}{\partial x_a}\{(\mathbf{K}\cdot \boldsymbol{u})\, u_a\} - \frac{\partial}{\partial t}(\mathbf{K}\cdot\mathbf{u}) . \quad (85)$$

All the terms in (84) and (85) can be interpreted physically without difficulty. In (84), the first three terms represent the Lorentz force and the action of the Joule heat. The fourth term is the divergence of the elastic and electromagnetic stress tensors. The next two terms are the rate of change of the momentum with which the medium is supplied by the elastic and electromagnetic stresses. These are relativistic terms. The remaining two terms are the rate of change of the momentum with which the medium is supplied by the electromagnetic fields. These also are relativistic terms.

Similarly, the terms in (85) can be interpreted. The first two terms represent both the work done by the Lorentz force per unit time and the Joule heat. The third term is the work done by the surface forces per unit time. The remaining four terms represent the rate of change of the energy with which the medium is supplied by the electromagnetic fields and the elastic stresses. These are relativistic terms.

In terms of f_i^+ defined by (83), the action of the electromagnetic fields and the elastic stresses on the media can be described by (82). However, $\mathbf{f}^+$ is not a mechanical force, because in general f_4^{+0} does not vanish. This fact can be seen also from the above interpretation of (84). The four-vector f_i^+ is the generalized four-force density.

VII. Conclusion

The electromagnetic fields supply the media with momentum and energy, because the electromagnetic fields interact with the media. In the total energy-momentum tensor, therefore, the electromagnetic part and the material part overlap each other. It is impossible to divide the total energy-momentum tensor into an electromagnetic tensor and a material tensor.

In a closed system, in general, only the total energy-momentum tensor has physical significance and can be defined.

Further, the generalized force which acts on the medium has been obtained and interpreted clearly.

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